Asymptotic estimates on the area complexity of lookup function

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***Abstract*—This article discusses the cellular circuit model of integrated circuits, which takes into account the physical features of their synthesis. The model differs from boolean circuit models in the additional requirements for circuit geometry, necessary for routing resources in ICs. The article explores the complexity of implementing specific systems and functions in the cellular circuit model. The study focuses on the complexity of implementing the lookup function, including asymptotically tight bounds for the area complexity of cellular circuits implementing it. The article concludes with constructive solutions to optimize the area complexity of cellular circuit multiplexers.**

***Index Terms*—cellular circuit, boolean circuit, lookup function, planar schemes, lower bounds, Thompson model, area complexity**

I. Introduction

Kravcov proposed for the first time the model of **cellular circuit** in "standart" basis of functional and commutation elements in 1967 [1]. There the **cellular circuit** complexity was understood as it's area. In general the **cellular circuit** of functional and commutation elements is a mathematical model of integrated circuits (IC), which takes into account the features of their physical synthesis. The fundamental difference of this model from the well-studied classes of boolean circuit (BC) is the presence of additional requirements to the circuit geometry, which cares necessary routing resources during creating an IC.

Albrecht showed in his work [2] that the Shannon function $A(n)$, characterizing the complexity of the most "complex" boolean function (BF) of n variables at $n = 1, 2, \dots,$ in the model [1] has asymptotically [1] tight bound $\sigma 2^n$, where σ is some constant. The exact value of the constant σ remains unknown, although it follows from works [2] and [1] that it is in the segment $[\frac{1}{4}, \frac{9}{2}]$. The cellular circuit basis, for which was succeeded to obtain the same asymptotic for the similar Shannon function with the constant $\sigma = 1$ was given by Gribok [3].

The asymptotically tight bound of the area complexity for some special BF and systems of BF were also described within the **cellular circuit** model. Shkalikova investigated [4] the implementation complexity by planar circuits of some BF specific systems, including systems of symmetric functions. She established that $Cn2^n$ is the asymptotically tight bound for the decoder complexity, i.e. a system of all 2^n elementary conjunctions of the rank n from n variables. The asymptotically tight bound n^2 for one specific Boolean function of n variables was received by Hromkovic, Lozhkin et al. [5]

The asymptotic tight bounds of some special BF or systems of BF complexity were investigated in a number of works, in the model of **cellular circuit** [6], [7], [8], and the model of the cellular contact circuits [9], [7], [10]. It was proved that for some BF systems type $F_n = (f_1, \dots, f_{N_n})$ of n variables, $n = 1, 2, \dots,$ the asymtotically tight bounds of complexity in those classes of circuits are equal to CnN_n, with stated constant C. For example, it turned out that for the decoder from n selector variables, for which $N_n = 2^n$, the constant C in both models is equal to $\frac{1}{2}$ etc. The phenomenon of antagonism [8] between the functional elements number and the cellular circuit area had been found for the specified decoder.

The questions of the implementation complexity in the cellular circuit model of functions from more "narrow", compared to the class of all functions, but, nevertheless, sufficiently "powerful" classes of functions, as well as partial functions, have been investigated. Zhukov proposed [11] a synthesis method of optimal in terms of the area power and depth of cellular circuits, implementing partial Boolean functions. The asymptotic kind of $C\frac{2^n}{logn}$ for an **cellular circuit** area with limited height and multiple inputs, implementing functions from a nonzero invariant class, where the constant C depends on that class power was established by Yablonskaya [12]

A similar mathematical model was described by Thompson in 1980 [13]. This model is fundamental for IC research. The fact that it does not take into account delays that occur during signal propagation is a significant disadvantage. Chazelle and Monier [14] proposed in 1985 an alternative model, correcting this deficiency. Bilardi, Pracchi and Preparata [15] studied in details the both models possibilities and their practical application. Their study showed that the Thompson's planar model is a satisfactory approximation for ICs, at least for single-chip systems. It remains an accurate approximation for IC small areas (individual components) in cases, where the model cannot correctly reflect all the designed systems features.

There are a number of works, including the aforementioned

[1]Let's say that consequences $a(n)$, $b(n)$, $n = 1, 2, \dots$ holds the asymptotic inequality $a(n) \lesssim b(n)$, if $a(n) \leqslant b(n)(1 + o(1))$. Moreover, the asymptotic equality $a(n) \sim b(n)$ is equivalent to that $a(n) \lesssim b(n)$ and $b(n) \lesssim a(n)$.

work [11], in which **cellular circuits** are optimized both by area and by some other parameters: depth, static power, average power, dynamic activity (power), et al. Cheremisin [16] investigated the measure of cellular circuits activity. He also showed that a decoder cannot be implemented by a circuit that is optimal in area and activity at the same time, in terms of the power of growth. Thereby their antagonism was revealed [8]. Kalachev [17], [18], [19] investigated the cellular circuits synthesis, implementing both random and partial BF, as well as BF from a special classes and on which the optimal in power of growth values of several of the complexity functionals (area, depth, cardinality) had been achieved for almost all implementable BF. Rybakov and Alehina [20] investigated the reliability of **cellular circuits**.

Lookup function l_n of n "selector" and 2^n "data" variables i.e. a so-called multiplexer BF of power n if often used in theoretical research and in IC synthesis. The BF value l_n is equal to it's data variable value, which number in the binary numeral system had entered the selector inputs.

Usually circuit implementation of that BF, called multiplexer, is a component part of more complex circuits, memory selection circuits and combinational blocks. Multiplexer BF are used both in the theory of individual synthesis when searching for optimal or close to optimal circuits and in the theory of universal synthesis when working-out a general method for circuit constructing and analyzing a Shannon function. Moreover, multiplexer type BF is used when testing and researching the circuits reliability.

Multiplexer BF implementation complexity became the subject of many authors study. Korovin [21] established an asymptotically tight bound type 2^{n+1} for complexity of the BF l_n both in the class of formulas and in the class of BC in standart basis was established. Rumiantsev [22] received asymtotically tight bounds type $2^{n+1} + \mathbf{O}(2^{\frac{n}{2}})$ for the complexity of BF l_n in the class BC over the standart basis. In the work [23] was considered asymptotically optimal by reliability implementation in the model of **cellular circuit** of the selector function v_{2i}, similar to the multiplexer BF.

Earlier in work [24] were received asymptotically tight bounds for the circuits area complexity, implementing the decoder of n selector variables. These estimates coincide in the decomposition first part, have the type $n2^{n-1}\left(1 \pm \mathbf{O}\left(\frac{1}{n}\right)\right)$ and can be considered asymptotic estimates of a high degree of accuracy as in work [25].

In this work we establish asymptotically tight bounds of the cellular circuit area complexity, implementing the lookup function of n selector variables.

II. Basic definitions and description of the model

The subjest of this work are **cellular circuits**. Each of them represents a rectangular lattice on a plane, consisting of cells – unit squares. Every cell is an element of the basis, which inputs and outputs would be located in the middle of it's sides contacts. Every contact would be either element input, or element output, or it's isolated pole. Every element of the basis is realizing the boolean functions system from boolean

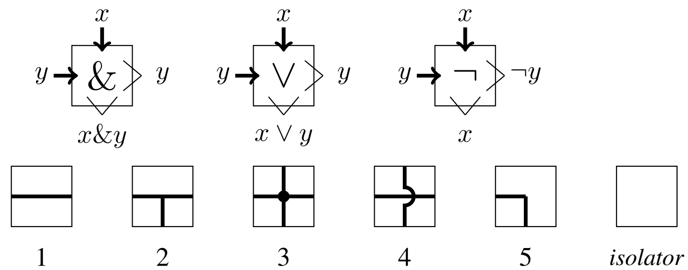

Fig. 1. Basis B_0. *Functional elements* (&) conjunction, ($\vee$) disjunction and ($\neg$) negation. *Commutation elements*: 1) wire, 2) T-junction, 3) X-junction, 4) crossing without junction, 5) turn. Here and then so-called axis of functional elements (vertical axis for signs & and $\vee$, horizontal axis for sign $\neg$) is co-directed to the nonequal exit.

variables on it's outputs, juxtaposed to it's inputs, with the exception of so-called *isolator*, an element all contacts of which are isolated poles.

Every element would be one of two types: commutation element (CE) and functional element (FE). The element would be called *functional* if it is implementing at least one non-identical, i.e. different from the variable, function. Any other element, including isolator, would be assumed *commutation* element. Thereby, commutation elements are realizing only identical functions and their purpose is to transmit signals. Let's assume that every FE of basis B would be realizing on it's outputs only one different from it's input boolean variables BF and let's call *basic* those outputs, on which it is realized.

In addition, the signal transmission can be realized through the FE input and such of it's outputs, on which would be realized BF, identically equal to that input, or otherwise, through the present input-output FE pair, which we would also call *commutation*. The commutation FE pair output and also every CE output would be assumed *commutation*.

In this paper the basis B_0 is considered one of the **cellular circuit** possible basis, related with standart basis of boolean algebra elements $B' = \{x_1 \wedge x_2, x_1 \vee x_2, \overline{x_1}\}$, that consists of 3 FE and 6 CE, including isolator (Pic. 1). Let's notice that every of three FE of that basis holds one input-output commutation pair. Let's remind that cellular circuit basis B_0, reviewed in papers [2], [1], [4], is also connected with basis B'.

The basis B_0 elements can be rotated by an angle multiple of 90° and also "flips" around any of the unit square symmetry axes while they are attaching into the rectangular lattice cells.

All inputs located on the **cellular circuit** rectangular lattice border are declared present **celullar circuit** *inputs*. Some outputs located on the cellular circuit rectangular lattice border are declared present **celullar circuit** *outputs*.

It is assumed that every **cellular circuit** Σ input is labelled by input boolean variable from a countable ordered alphabet $\mathfrak{X} = \{x_1, \ldots, x_n, \ldots\}$. It is assumed that every **cellular circuit** Σ output is labelled by output boolean variables set from a countable ordered alphabet $\mathfrak{Z} = \{z_1, \ldots, z_m, \ldots\}$.

When describing the cellular circuit functioning let's in accordance with [**?**] label through $P_2(n)$ the set of all boolean functions f from boolean variables $X(n) = \{x_1, \ldots, x_n\}$, every of which is a relation type $B^n \xrightarrow{f} B$, where B^n is a n-th Cartesian power of a set $B = \{0, 1\}$, or in other words, unit n-dimensional cube. Herewith m power $P_2^m(n)$ of the set $P_2(n)$ would consist of so-called (n, m)-operators, i.e. of systems $F = (f_1, \ldots, f_m)$, where $f_i \in P_2(n)$ for all $i, i = 1, \ldots, m$.

For the structure description of **cellular circuit** $\Sigma = \Sigma(x_1, \ldots, x_n; z_1, \ldots, z_m)$ i.e. **cellular circuit** with input bollean variables $X(n) = \{x_1, \ldots, x_n\}$ and output bollean variables $Z(m) = \{z_1, \ldots, z_m\}$ in the basis B_0, connected with basis B', let's compare Σ with an acyclic labelled oriented graph G such that:

1) set of vertices G one-to-one corresponds to the functional elements set and inputs Σ, moreover the vertex label is either the type corresponding FE or the input BV corresponding to input of ;
2) arc (u, v) is an arc of G then and only then, when from associated with u "node", i.e. the corresponding to it FE main output or input of Σ, through the chain of connected in series commutation input-output elements pairs Σ can be reached the input of FE, corresponding to vertex v of G;
3) the vertex u of graph G is labelled as output BV z_j then and only then, when either z_j occurs to be label of corresponding u node Σ, or from the specified node Σ can be reached element Σ commutation output, labelled z_j in the sense of previous paragraph.

Let's assume that constructed in such a way graph is a boolean circuit $S = S(x_1, \ldots, x_n; z_1, \ldots, z_m)$ in the basis B', and consider that functioning S, which is given by boolean functions system $F = (f_1, \ldots, f_n)$ from $P_2^m(n)$, also determines functioning of **cellular circuit** Σ.

Let's introduce cellular circuit *area complexity*, that would further be their complexity measure. The circuit Σ, which contains no rows or columns, consisting only of isolators, has such dimensions: *length* $l(\Sigma)$, measured horizontally, and *height* $h(\Sigma)$, measured vertically. Everywhere further we would assume that $h(\Sigma) \leqslant l(\Sigma)$ without loss of generality. The rectangular lattice area of the cellular circuit Σ, i.e. multiplication of it's length $l(\Sigma)$ to the height $h(\Sigma)$ is called the area $A(\Sigma)$ of cellular circuit Σ and, thus,

$$A(\Sigma) = l(\Sigma)h(\Sigma). \tag{1}$$

The value $A(F)$ for BF system $F = (f_1, \ldots, f_m)$ from $P_2^m(n)$, would be defined equal to minimum area of cellular circuits realizing F, which would be called the system F area complexity.

Let's notice some other **cellular circuit** features which are important for the construction. A circuit with all input variables on only one side is called one-sided. The circuit is called *repeating* (an input x) circuit if it has symmetrically locating output $y(x) = x$ opposite to every input x.

III. Multiplexer and upper bound for it's area

Let's remind that the standart *multiplexer* from n selector variables or, in other words, lookup function l_n is a boolean

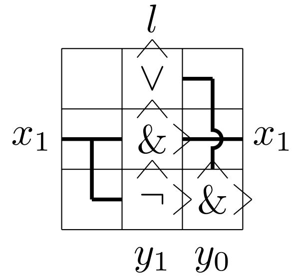

Fig. 2. Cellular circuit $\hat{M}_1$, implementing multiplexer from $n = 1$ selector variables.

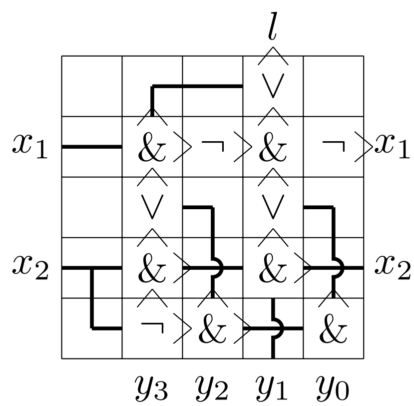

Fig. 3. Cellular circuit $\hat{M}_2$, implemening multiplexer from $n = 2$ selector variables.

function l_n with n selector inputs $x_1, \ldots, x_n$ and 2^n data inputs $y_0, \ldots, y_{2^n-1}$. This function is identically equal to data input $y_{\nu(\sigma)}$, i.e. data input with the number $\nu(\sigma) = \sum_{i=1}^{n} \sigma_i 2^{n-i}$, at the tuple $\sigma = (\sigma_1, \ldots, \sigma_n) \in B^n$ on the selector inputs. In other words, lookup function (multiplexer BF) can be represented as the following DNF

$$l_n(x_1, \ldots, x_n, y_0, \ldots, y_{2^n-1}) = V_{\sigma=(\sigma_1,\ldots,\sigma_n)\in B^n} x_1^{\sigma_1} \cdots x_n^{\sigma_n} y_{\nu(\sigma)},$$

where, as always, $x_i^0 = \overline{x_i}, x_i^1 = x_i$.

Lemma 1:

There is a **cellular circuit** $\hat{M}_n$ of height $2n+1$ and length $2^n + 1$, realizing BF l_n with repetition of its selector inputs, located only on one of it's vertical sides, and allocation of it's data inputs on one of it's horizontal sides, and the output on another horizontal side.

a) Proof.: Let's construct the circuit S by induction on $n = 1, 2, \ldots$. All input variables would be given from the left side. The variable x_n is separated in the first column into two rows, through one of them is transferring the variable itself, and through the other one is transferring the function equal to this variable negation. Data variables y_j, $j = 0, \ldots, 2^n - 1$ are located in sequence on the circuit bottom side. They go through the columns from bottom to top, passing through the vertical main input-output pairs of conjunction elements and commutation input-output pairs of negation elements.

Induction basis Let's realize lookup function $l_1(x_1, y_0, y_1)$ and $l_2(x_1, x_2, y_0, y_1, y_2, y_3)$ in a way shown on the pic. 2 and 3.

Induction step For the $p \geqslant 3$ let it be constructed a scheme $\hat{M}_{p-1}$ for $p-1$ selector inputs $x_2, \ldots, x_p$ with the size $2^{p-1}+1$ in length and $2p-1$ in height. Let's compose two present circuits consequentially in such a way that adress variables repeating outputs of one circuit would be inputs of another circuit, and data inputs would be located on the same side. By the theorem condition that subcircuits according outputs z_1 and z_2 are located on the top side.

Let's delete second subscheme's first column, realizing variable partition x_p. Let's replace in the first subcircuit bottom row the last conjunction element (with isolated output) with conjunction element repeating it's input at the output, so realizing according variable ($\overline{x_p}$) negation in the bottom row. Thus the common circuit length would be $2^{p-1} + 1 + 2^{p-1} + 1 - 1 = 2^p + 1$.

Let's add a selector variable x_1 input on the circuit left side in the new top row. Let's hold a commutation lines row from the left to the right, setting the conjunction elements on outputs z_1 and z_2 so that conjunction $x_1 z_1$, $\overline{x_1} z_2$ outputs would be on the circuit top side. Let's hold a negation element after every such conjunction, providing the variable x_1 negation for the start and next returning to the initial variable at the output, repeating it's selector variable input. The circuit length for $p - 1 \geqslant 2$ is greater or equal 3 by condition, therefore it is possible to hold in the row at least 2 elements opposite each circuit. Let's add a row from the turn, commutation lines and disjunction in such a way that the turn would be located opposite conjunction output, realizing $x_1 z_1$. Corresponding value would be held along the row lines to disjunction element, on which second input is located conjunction output, realizing $\overline{x_1} z_2$, and disjunction output would be directed to the circuit top side.

Thereby a function $x_1 z_1 \vee \overline{x_1} z_2$ is realized on the the constructed circuit top side S, where z_1, z_2 are multiplexer outputs from $p-1$ selector variables $x_2, \ldots, x_p$ and relevant data variables. It means that this circuit is realizing multiplexer from p selector variables $x_1, \ldots, x_p$ and relevant data variables by construction. Selector variables inputs of S are located on the circuit left side, and data variables inputs are located on the bottom. Main output of the circuit S is located on it's top side, and supporting outputs, repeating selector variables inputs, on the right side.

Following equalities would be held by construction in this case:

$$l(S) = 2^p + 1, \quad h(S) = 2p + 1.$$

and, consequently, assuming $S = \hat{M}_p$, we would get the desired circuit which at $p = n$ satisfies the lemma's conditions. $\square$

Theorem 1 (about the upper bound): There is a circuit M_n, realizing multiplexer BF l_n, with the area

$$A(M_n) = (n + 10)2^{n-1} + 2n + 30$$

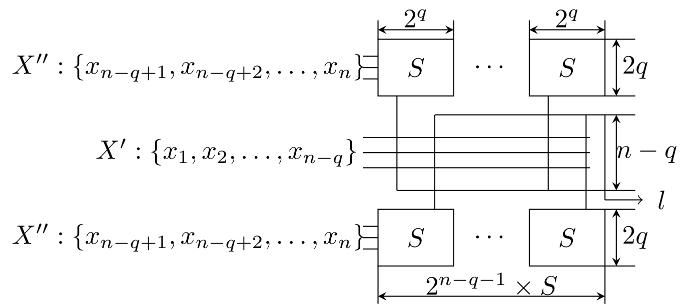

Fig. 4. Circuit M_n general view. Scheme data inputs suitable for blocks S are not shown here.

b) Proof.: Let the selector boolean variables set $X = X(n)$ of function l_n be separated into two sets X' and X'', accordingly consisting of $n - q$ and q variables, $q > 1$, i.e.

$$X = X' \cup X'', \ X' \cap X'' = \emptyset, \tag{2}$$
$$|X'| = n - q, \ |X''| = q. \tag{3}$$

Let's consider that

$$X' = \{x_1, \ldots, x_{n-q}\}, X'' = \{x_{n-q+1}, \ldots, x_n\},$$

and that the data variables set $y = (y_0, \ldots, y_{2^n - 1})$ of l_n function has separation into 2^{n-q} consecutive subsets $y^{(0)}, \ldots, y^{(2^{n-q}-1)}$, each of those consists of 2^q data variables with consecutive numbers.

Let's construct for the input set X'' two rows, consisting of 2^{n-q-1} circuits S type $\hat{M}_{n-q}$ each, and realizing multiplexers from 2^q data BV set $y^{(i)}, i = 0, \ldots, 2^{n-q} - 1$. One row is located on the circuit top side, another on the bottom side. Let's lengthen their outputs in such a way that they would construct a "comb". Let's compose two such rows opposite each other by combining "combs". Let's direct the rows, passing variables from the set X', perpendicular to these outputs. Such a lattice would have one disjunction in every crossing. By using necessary number of negations in the space between vertical rows would be changed the tuples of variables from the set X', (see pic. 4, 5). As a result every multiplexer S output is passing through conjunctions with the set of all tuples of variables from X' in some degrees $\sigma_1, \ldots, \sigma_{n-q}$, as the result of which for $i = 0, \ldots, 2^{n-q} - 1$ would be realized the BF:

$$f_i(x'', y^{(i)}) = l_q(x_{n-q+1}, \ldots, x_n, y^{(i)}) x_1^{\sigma_1} x_2^{\sigma_2} \cdots x_{n-q}^{\sigma_{n-q}}.$$

Two rows of such outputs are connected by disjunctions into two lines. They are connected through the vertical conductor on the circuit right side. Another one disjunction is located in the place of connection which is realizing the lookup function on it's output.

The circuit necessary part is spreading width q in the circuit left part. It duplicates variables from the set X'' to the scheme top part.

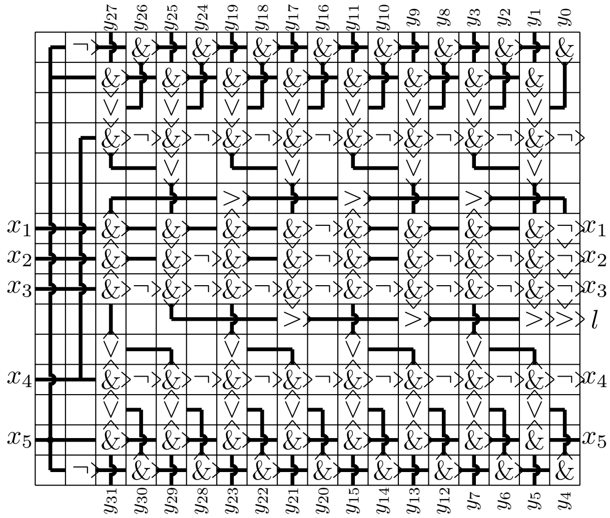

Fig. 5. View of cellular circuit M_5 with $q = 2$

For every tuple σ' of variables values from X' the signal of only one block type S reaches the output z. Every block is realizing a multiplexer of order q. All the scheme is realizing the multiplexer BF l_n by construction.

Let's estimate it's area:

$$l(S) = \frac{1}{2}2^{n-q} \cdot 2^q + q$$

$$h(S) = n - q + 2 \cdot (2q + 1) + 2$$

$$A(S) = l(S) \cdot h(S) = (2^{n-1} + q) \cdot (n + 3q + 4).$$

By taking small enough $q = 2$ could be get a circuit with the area

$$A(S) = (n + 10)2^{n-1} + 2n + 30,$$

where "the main" term is – $n2^{n-1}$.

□

Corollary 1: By constructing the circuit from theorem 1 for every $q = o(n)$ and $n = 1, 2, \ldots$ is true that

$$A(S) = n2^{n-1} + \mathbf{O}(q2^n).$$

Corollary 2: For $n = 1, 2, \ldots$ holds the inequality

$$A(l_n) \leqslant n2^{n-1} + \mathbf{O}(2^n).$$

IV. The lower bound of area complexity

Let vertical cut π be dividing **cellular circuit** Σ into two parts, the left circuit Σ' and the right circuit Σ''. Let tuples x', x'' be tuples accordingly composed from **BV** sets X' and X'' into which the cut π separates the input set **BV** X of **cellular circuit** Σ. Herewith $X = X' \cup X''$, $X' \cap X'' = \varnothing$, $|X'| = n'$, $|X''| = n''$, $|X| = n = n' + n''$. The ordered tuple of pole values, i.e. inputs-outputs Σ' and Σ'', located on π (see. pic. 6) would be called *the state* $\pi(\alpha', \alpha'')$ of the cut π, where $\alpha' \in B^{n'}$ and $\alpha'' \in B^{n''}$.

Herewith *table state* of the cut π of the circuit Σ for naturally ordered tuples $\alpha' \in B^{n'}$ and $\alpha'' \in B^{n''}$ would be called matrix $M \in B^{n' \times n''}$ with n' rows and n'' columns, for which $M_{ij} = \pi(\alpha', \alpha'')$, where $dec(\alpha') = i - 1$, $dec(\alpha'') = j - 1$ for all $i = \overline{1, 2^{n'}}$, $j = \overline{1, 2^{n''}}$.

In Shkalikova work [4] was proved that in such a case from the equality $\pi(\alpha', \alpha'') = \pi(\beta', \beta'')$ follows that

$$\pi(\alpha', \alpha'') = \pi(\alpha', \beta'') = \pi(\beta', \alpha'') = \pi(\beta', \beta'') \qquad (4)$$

It could be seen that for the BF system $F(x', x'')$, realized by **cellular circuit** Σ, would be true equalities

$$F(\alpha', \alpha'') = F(\alpha', \beta'') = F(\beta', \alpha'') = F(\beta', \beta''),$$

if every curcuit Σ output, located in Σ', has a duplicate in Σ'' and vice versa.

Lemma 2:

Let $\mathcal{I}$ and $\mathcal{J}$ be sets of those rows and, accordingly, columns of the state table M of the cut π of **cellular circuit** Σ, in which the same state q occurs at least once. Then all the submatrix of the matrix M, composed of rows with numbers from $\mathcal{I}$ and columns with numbers from $\mathcal{J}$, consists only of the states q.

a) Proof.: Let the lemma statement be false. Then there exists such a row $i, i \in \mathcal{I}$ and a column $j, j \in \mathcal{J}$ that $M_{ij} \neq q$. It is assumed that i and j would be included in the present matrix, i.e. there exists $k' \in \mathcal{I}$, $k'' \in \mathcal{J} : M_{ik''} = q$, $M_{k'j} = q$.

From the equality (4) directly follows by corresponding values substituting that

$$\pi(\nu^{-1}(i), \nu^{-1}(k'')) = \pi(\nu^{-1}(i), \nu^{-1}(j)) =$$
$$= \pi(\nu^{-1}(k'), \nu^{-1}(k'')) = \pi(\nu^{-1}(k'), \nu^{-1}(j)).$$

Hence appears a contradiction that lays in the fact that

$$q \neq \pi(\nu^{-1}(i), \nu^{-1}(j)) = \pi(\nu^{-1}(k'), \nu^{-1}(j)) = q.$$

The cotradiction appears from the assumption that there exists a row i and a column j such that $M_{ij} \neq q$. Therefore the assumption is not true and the lemma statement is true.

□

Lemma 3: Let **cellular circuit** Σ of hight h and length l, where $h \leqslant l$, from **BV** $X(n) = \{x_1, \ldots, x_n\}$ and $Y(n) = \{y_0, \ldots, y_{2^n-1}\}$ be realizing BF $l_n(X, Y)$. Then the height of the circuit Σ for large enough n satisfies the inequality

$$h(\Sigma) > n - 2 - 2\log n.$$

b) Proof.: Let π be a Σ cut, which separates the set BV $X(n)$ ($Y(n)$) into the right X'' (accordingly Y'') and the left X' (accordingly Y') subsets. Herewith $|X'| = n'$, $|X''| = n''$, $|Y'| = m'$, $|Y''| = m''$ would be such that $m' \geqslant 2^{n-1} - 1$ and $m'' \geqslant 2^{n-1} - 1$.

Let π' (accordingly π'') be such a part of the cut π, that is connected with outputs of elements Σ'' (accordingly Σ') and let $|\pi'| = s', |\pi''| = s''$ (see pic. 6). Let's assume that

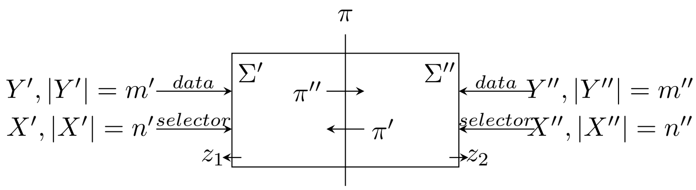

Fig. 6. To the proof. Here $|\pi'| = s'$, $|\pi''| = s''$.

the output $z = z_1$ of **cellular circuit** Σ, located in one of it's parts, has a "duplicate" z_2 in another part Σ. For such a construction would be enough to increase the height of Σ not more than 1.

Let's notice that the number of those tuples kind (α', α''), where $\alpha' \in B^{n'}$ and $\alpha'' \in B^{n''}$, for which $y_{\nu(\alpha',\alpha'')} \in Y'$, is equal to m'. Thus the average value of this BV number over all tuples $\alpha', \alpha' \in B^{n'}$ would be not less than $\frac{m'}{2^{n'}}$. Therefore it could be found such a tuple $\hat{\alpha}', \hat{\alpha}' \in B^{n'}$, for which the number of incoming into the Y' BV type $y_{\nu(\hat{\alpha}',\alpha'')}$, where $\alpha'' \in B^{n''}$ would be not less than $\left\lceil \frac{m'}{2^{n'}} \right\rceil \geqslant 2^{n''-1}$. Let's denote the set BV type $\hat{Y}'$, and the set of corresponding tuples α'' from $B^{n''}$ as A'', and assume that

$$\left|\hat{Y}'\right| = |A''| = \hat{m}' \geqslant 2^{n''-1} \tag{5}$$

Let's fixate the BV values from X' by the tuple $\hat{\alpha}'$, and the BV values from sets $Y' \backslash \hat{Y}'$ and Y'' by zeros. Then let's construct matrix M with $2^{\hat{m}'}$ rows and $\hat{m}'$ columns, which is a part of the *table state* of the cut π of the circuit Σ, which rows, taking into account given above fixation, are related with set of tuples $B^{\hat{m}'}$ of BV values from $B^{\hat{m}'}$, and columns with set A'' of tuples of BV values from X''.

Consider a random row S of the matrix M, coinciding the tuple $\hat{\beta}'$ from cube $B^{\hat{m}'}$ from boolean variables $\hat{Y}'$, i.e. let's fixate all inputs Σ', which are inputs Σ, by the tuple $(\hat{\alpha}', \hat{\beta}')$. Let's notice that *the state* of π'' in the row S which is a part of the cut π depends only from inputs π' of the scheme Σ'. Therefore the number of the cut π differently states in S would be not more than $2^{s'}$. It means, taking into account (5), that in S could be found such a *state* γ of cut π that occurs in S at least $2^{n''-s'-1}$ times. Such a *state* γ would be assumed a *dedicated state* of S. Let's notice that the cut π number of various values is less or equal $2^{s'+s''}$. Therefore could be found matrix M number of rows more or equal to $2^{\hat{m}'-s'-s''}$, each row consists the same dedicated state γ_0.

From the lemma 2 follows that there is a submatrix $\tilde{M}$ in M with l, $l \geqslant 2^{m'-s'-s''}$ rows and k, $k \geqslant 2^{n''-s'-1}$ columns, consisting of the states γ_0. Assume wherein δ_0 would be the value of such a position γ_0, that corresponds with the output **cellular circuit** Σ value, transferred from one part of Σ to another. By construction the column with number i, $i = 1, \ldots, k$ of the matrix M is related to some tuple α''_i from A'' and by that it corresponds to boolean variable y'_i from $\hat{Y}'$ such that $y'_i = y_{\nu(\hat{\alpha}',\alpha''_i)} = l_n(\hat{\alpha}', \alpha''_i, y_0, \ldots, y_{2^n-1})$. Therefore the values of all boolean variables $y'_1, \ldots, y'_k$ in any tuple $\hat{\beta}'$, corresponding to the row $\tilde{M}$, would be equal to each other and equal to δ_0. It means, that the matrix $\tilde{M}$ lines number would be less or equal to the number of those tuples $\hat{\beta}'$, $\hat{\beta}' \in B^{\hat{m}'}$ for which would be held the equality $y'_1 = \cdots = y'_k = \delta_0$, that is less or equal than $2^{\hat{m}'-k}$. Thus, the inequality holds:

$$2^{\hat{m}'-s'-s''} \leqslant 2^{\hat{m}'-2^{n''-s'-1}},$$

from which follows that

$$s' + s'' \geqslant 2^{n''-s'-1}. \tag{6}$$

Swapping the parts Σ' and Σ'' in the same way could be shown that

$$s' + s'' \geqslant 2^{n'-s''-1}. \tag{7}$$

Multiplying (6) and (7), would be obtained the inequality

$$2^{s'+s''}(s' + s'')^2 \geqslant 2^{n-2},$$

from which follows that

$$s' + s'' \geqslant n - 2 - 2\log n.$$

The statement of this theorem comes from the fact that taking into account duplicated outputs Σ, $h+1 \geqslant s = s' + s''$:

$$h \geqslant n - 3 - 2\log n.$$

□

Theorem 2 (about the lower bound): For multiplexer BF l_n, over $n = 1, 2, \ldots$, wouid be true the lower bound:

$$A(l_n) \geqslant (n - 3 - 2\log n)2^{n-1} + \mathbf{O}(n^2).$$

c) Proof.: Assume that Σ is a (single) **cellular circuit** implementing the multiplexer l_n with length l and height h. Without loss of generality we consider $l \geqslant h$.

Let's impose restrictions on the circuit Σ perimeter, proceeding from circuit inputs and outputs total number:

$$2(h + l) \geqslant n + 2^n + 1, \tag{8}$$

from which follows that:

$$l \geqslant \frac{1}{2}(n + 2^n + 1) - h$$

Herewith the circuit area Σ satisfies the correlations:

$$A(\Sigma) = l \cdot h \geqslant h\Big(\frac{1}{2}(n + 2^n + 1)\Big) - h^2$$

It is obvious that the problem of area minimizing solution $A(\Sigma)$ with restriction $h \leqslant l$ consists in the value minimum h. By the lemma 3 that minimum is more or equal to $n - 2 - 2\log n$. Accordingly, over $n \to \infty$ we obtain the inequality of the theorem

$$A(\Sigma) \geqslant (n - 3 - 2\log n)2^{n-1} + \mathbf{O}(n^2)$$

$\square$

Corollary 3:

From the theorem 2 over $n = 1, 2, \ldots$ follows asymptotic inequality

$$A(l_n) \gtrsim n2^{n-1}.$$

Corollary 4:

From the theorem 1 and the theorem 2 follows that for all $n = 1, 2, \ldots$ in the model **cellular circuit** are true asymptotically tight bounds

$$n2^{n-1} - \mathbf{O}(\log n2^n) \leqslant A(l_n) \leqslant n2^{n-1} + \mathbf{O}(2^n).$$

Corollary 5:

From the corollary 4 follows that if $n \to \infty$ then asymtotic of area complexity is

$$A(l_n)\ \frac{1}{2}n2^n\big(1 \pm \mathbf{O}(\frac{\log n}{n})\big),$$

which can be considered close to the asymptotically tight bounds of a high degree of accuracy [25].

V. Conclusion

In this paper we showed asymptotically tight bounds for the area complexity. The higher bounds was achieved by constructively building a family of cellular circuit multiplexers from n selector variables. The lower bounds were, as usual, determined from information theory considerations. Moreover, obtained tight bounds coincide with an accuracy up to a constant in the leading term of the decomposition and degree of the second one. They are close to asymptotically tight bounds of a high degree of accuracy.

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